



**Australian Government**  
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Defence Science and  
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# **Stochastic Representations of the Marcum Q-Function and Associated Radar Detection Probabilities**

***Graham V. Weinberg***

**Electronic Warfare and Radar Division  
Defence Science and Technology Organisation**

**DSTO-RR-0304**

## **ABSTRACT**

This report introduces new research on the generalised Marcum Q-Function, and in particular, the probability of detection of a number of incoherently integrated signals in a Gaussian clutter and noise environment. A new probabilistic association is derived, linking this detection probability with a probability associated with two independent Poisson random variables. Additionally, it is shown that this detection probability is the solution to two stochastic Volterra integral equations. This results in a means of obtaining estimates of this detection probability. Specifically, lower and upper bounds are derived using these representations, and the bounds are compared with known results. As a by-product of this work a new useful expression for the differences in distributions of independent Poissons random variables is obtained.

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DSTO-RR-0304

*Published by*

*Defence Science and Technology Organisation*

*PO Box 1500*

*Edinburgh, South Australia, Australia 5111*

*Telephone: (08) 8259 5555*

*Facsimile: (08) 8259 6567*

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*AR No. AR-013-559*

*December, 2005*

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# Stochastic Representations of the Marcum Q-Function and Associated Radar Detection Probabilities

## EXECUTIVE SUMMARY

The work presented here is a consequence of the ongoing long range research associated with radar detection issues that arose out of the task AIR 01/217, which has now been succeeded by AIR 04/216. The purpose of this task is to provide the Royal Australian Air Force with technical advice on the performance of the Elta EL/M-2022 maritime radar. The latter radar is used in the AP-3C Orion fleet. Key performance measures of a radar include probabilities of false alarm and detection. Earlier work by the author focused on how Monte Carlo methods could be used to estimate such quantities. The research found in this report arose out of the need to find an efficient Monte Carlo estimator for a single pulse probability of detection of a target in Gaussian clutter and noise. This was a component in a two-tiered Monte Carlo estimator for a binary integrated detection probability. A new association was discovered between the pulse detection probability and a pair of independent Poisson random variables. The mathematics behind this discovery is presented here. Furthermore, it turns out that this result can be extended to the case of a series of incoherently integrated pulses. Consequently, this provides a new twist on the generalised Marcum Q-Function. The latter is an important function in the study of radar and communications, and hence new representations for it are of interest to the wider radar community.

In addition to presenting this new result, it is shown that this detection probability is the solution to two stochastic Volterra integral equations of the second kind. These representations for the detection probability suggest ways in which bounds can be obtained. The importance of bounds on such probabilities is that they indicate the minimum and maximum performance levels of a radar detection scheme. We construct new lower and upper bounds on a specific detection probability, and compare them with well-known results from the signal processing literature.



## Author

**Graham V. Weinberg**

*Electronic Warfare and Radar Division*

Graham V. Weinberg is a specialist in mathematical analysis and applied probability, and has been a DSTO radar research scientist since early 2002. His interests are in the mathematics of radar signal processing.

In 2001 he graduated with a Doctor of Philosophy Degree from The University of Melbourne, in the area of probability approximations with Stein's Method. He has had experience in both the academic and industrial arenas, including employment in finance and telecommunications.

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# Contents

<b>Glossary</b>	<b>xi</b>
<b>1 Preliminaries</b>	<b>1</b>
1.1 Background . . . . .	1
1.2 Probability of Detection in Gaussian Clutter . . . . .	1
1.3 The Marcum Q-Function . . . . .	3
1.4 Contributions of this Report . . . . .	4
<b>2 Marcum's Q-Function and a Poisson Connection</b>	<b>5</b>
2.1 Generalised Marcum Q-Function . . . . .	5
2.2 The Poisson Connection . . . . .	5
<b>3 Stochastic Representation</b>	<b>8</b>
3.1 Stein's Method for Poisson Approximation . . . . .	8
3.2 Differences of Poisson Distributions . . . . .	9
3.3 Stochastic Representation I . . . . .	11
3.4 Stochastic Representation II . . . . .	13
<b>4 Bounds on the Pulse Detection Probability</b>	<b>15</b>
4.1 Lower and Upper Bounds . . . . .	15
4.1.1 Case 1: $\varsigma > \tau$ . . . . .	15
4.1.2 Case 2: $\varsigma < \tau$ . . . . .	16
4.2 Some New Lower and Upper Bounds . . . . .	17
4.2.1 Case 1: $\varsigma > \tau$ . . . . .	17
4.2.2 Case 2: $\varsigma < \tau$ . . . . .	18
4.2.3 Some Comments on the Representation in Equation (64) . . . . .	19

4.3	Numerical Comparison of Bounds . . . . .	19
<b>5</b>	<b>Conclusions and Future Directions</b>	<b>30</b>
<b>6</b>	<b>Acknowledgements</b>	<b>30</b>
	<b>References</b>	<b>31</b>
	<b>Appendix A: Proof of Equation (53)</b>	<b>33</b>



# Figures

1	Comparison of upper bounds for the case where $\varsigma > \tau$ , showing the bounds (54) and (70) as functions of $\tau$ and $\varsigma$ . ‘C-F’ refers to the upper bound due to Corazza and Ferrari, namely (54), while ‘Wei’ is the upper bound in (70). The vertical scale is in logarithms. . . . .	20
2	Similar plot to that of Figure 1, except on a different range of $\varsigma$ and $\tau$ values.	21
3	A comparison of upper bounds in the case where $\varsigma < \tau$ . The new upper bound in (76), labelled ‘Wei’, is outperformed by the bound (62) due to Chiani, labelled as ‘Chi’. . . . .	22
4	The same plot as for Figure 3, with the removal of the two worst performing bounds. . . . .	23
5	Comparison of upper bounds in the case where $\varsigma < \tau$ , but over a different range of $\varsigma$ and $\tau$ parameters. In both subplots, it is clear that the bounds ‘Sim’, due to Simon (63) and ‘S-A’, due to Simon and Alouini (61) are unsatisfactory. Figure 6 shows the same plot with these two upper bounds removed.	24
6	This is the same as Figure 5, except the two worst performing bounds have been removed. It is now clear that the bound labelled ‘C-F’, due to Corazza and Ferrari (60) is also unsatisfactory. The bound due to Chiani (62) performs the best, while the new upper bound in (76), labelled as ‘Wei’, is the second best. . . . .	25
7	Plots of lower bounds in the case where $\varsigma > \tau$ . The same labelling convention has been employed as in previous figures. The exact value, labelled as ‘Exact’, is almost on the horizontal axis in each subplot. It is clear that the new bound (70) is performing very well. See Figure 8 for clarification of this. . . . .	26
8	The same plot as for Figure 7, except with the removal of the bound (59) due to Simon. As for Figure 7, the exact value is almost on the horizontal axis. The best bound is that due to Corazza and Ferrari (56), while the new lower bound in (70), namely ‘Wei’, is second best. . . . .	27
9	Lower bounds for the case where $\varsigma < \tau$ . This is an example where the new lower bound in (76) has very good performance. The only bound that is comparable to the new lower bound is that due to Chiani (62). The latter is better in the second subplot when $\varsigma$ is very small. . . . .	28
10	The same plot as in Figure 9, with the inclusion of the Chiani lower bound in (62), and the new lower bound in (76), and the exclusion of the other lower bounds. The first subplot shows the improvement more clearly, while the second subplot shows there is a point where the better bound switches from the Chiani one to the new lower bound. . . . .	29



## Glossary

$\mathbb{N}$  Natural numbers  $\{0, 1, 2, \dots\}$ .

$\mathbb{P}$  Probability.

$\mathbb{E}$  Statistical expectation.

$\mathbb{I}$  Indicator function:  $\mathbb{I}_{[x \in A]} = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$

$:=$  Defined to be.

$\varsigma$  Signal to noise ratio (SNR).

$\tau$  Detection threshold.

$I_n(x)$  Modified Bessel function of order  $n$ :  $I_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (-ie^{-i\theta})^n e^{-x \sin \theta} d\theta$ .

$Q_n(\alpha, \beta)$  Generalised Marcum Q-Function:  $Q_n(\alpha, \beta) = \frac{1}{\alpha^{n-1}} \int_{\beta}^{\infty} x^n e^{-\left(\frac{x^2 + \alpha^2}{2}\right)} I_{n-1}(\alpha x) dx$ .

$\rho_n(\varsigma, \tau)$  Detection probability of  $n$  incoherently integrated signals:  $\rho_n(\varsigma, \tau) = Q_n(\sqrt{2n\varsigma}, \sqrt{2\tau})$ .

$\rho(\varsigma, \tau) = \rho_1(\varsigma, \tau)$ .

**Erfc**( $z$ ) Complementary error function:  $\mathbf{Erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt$ .

$a \wedge b$  Minimum of  $a$  and  $b$ .

$a \vee b$  Maximum of  $a$  and  $b$ .

$\stackrel{d}{=}$  Equality in distribution:  $X \stackrel{d}{=} Y$  is equivalent to  $\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$  for all sets  $A$ .

**Po**( $\lambda$ ) Poisson Distribution with mean  $\lambda > 0$ : if  $X \stackrel{d}{=} \mathbf{Po}(\lambda)$ , then  $\mathbb{P}(X = j) = \frac{e^{-\lambda} \lambda^j}{j!}$ , for all  $j \in \mathbb{N}$ .

**Po**( $\lambda$ ){ $A$ } Cumulative Poisson probability on set  $A \subset \mathbb{N}$ :  $\mathbf{Po}(\lambda)\{A\} = \sum_{j \in A} \frac{e^{-\lambda} \lambda^j}{j!}$ .

**Bin**( $n, p$ ) Binomial Distribution with parameters  $n \in \mathbb{N}$  and  $p \in [0, 1]$ : if  $X \stackrel{d}{=} \mathbf{Bin}(n, p)$ , then  $\mathbb{P}(X = j) = \binom{n}{j} (1 - p)^{n-j} p^j$ , for  $j \in \{0, 1, \dots, n\}$ .

**R**( $\alpha, \beta$ ) Uniform (or rectangular) distribution on the interval  $[\alpha, \beta]$  ( $\alpha < \beta$ ): If  $X \stackrel{d}{=} \mathbf{R}(\alpha, \beta)$ , then  $\mathbb{P}(X \leq x) = \frac{x - \alpha}{\beta - \alpha}$ , for  $x \in [\alpha, \beta]$ .

# 1 Preliminaries

## 1.1 Background

The single pulse probability of detection, for a constant target in Gaussian clutter and noise, has been the subject of much investigation in foundational studies in the mathematical analysis of radar detectors. Expressions for such probabilities first appeared in [Rice 1944 and 1945, Marcum 1960 and Di Franco and Rubin 1968], and more recently, [Levanon 1988 and Minkler and Minkler 1990] contain detailed analyses deriving this classical detection probability.

Under a Neyman-Pearson regime, this probability of detection appears as an integral involving a modified Bessel function of order zero. This integral does not have a closed analytic form. As it appears in a number of modelling applications, such as binary integration [Shnidman 1998], a number of authors have examined ways of estimating it efficiently. [Shnidman 1995, 1998 and 2002] applied a Maclaurin series expansion to the Bessel function, and consequently obtains a number of useful results. Simulation methodology can also be used to estimate this pulse probability of detection. In [Weinberg and Kyprianou 2005], Monte Carlo methods are used to estimate it.

As a result of searching for an efficient estimator for this pulse detection probability, a new probabilistic expression for it has been obtained. It can be shown [Weinberg and Kyprianou 2005] that the detection probability is identical to a comparison of two independent Poisson random variables. One is centred on the average signal strength, while the other is centred on a normalised detection threshold.

## 1.2 Probability of Detection in Gaussian Clutter

The following is included for completeness, and to provide a context for the work to be discussed in subsequent sections, and is based upon the developments in [Levanon 1988]. Consider a radar operating in Gaussian clutter and noise, from which a single pulse is transmitted. The transmitted signal is a sine wave with period  $\psi$  and frequency  $\omega$ . The returned signal will be assumed to be a phase shifted version of the original, with the addition of interference. For modelling simplicity, we do not differentiate between clutter and noise, nor any other environmental factors that may distort the signal. We will just assume the total interference is a Gaussian random variable. The returned signal is passed to a narrow bandpass filter, with centre frequency  $\omega$ . We assume this filter has a rectangular response with bandwidth  $f_B$ . Then assuming that  $f_B > \frac{1}{\psi}$ , the returned signal is

$$s(t) = A \cos(\omega t - \theta) = a \cos(\omega t) + b \sin(\omega t), \quad (1)$$

where  $\theta = \arctan\left(\frac{b}{a}\right)$  is the phase shift of the signal, and the amplitude is  $A = \sqrt{a^2 + b^2}$ . We assume that  $a$  and  $b$ , and consequently  $A$ , are deterministic, and that  $\theta$  is uniformly distributed on the interval  $[0, 2\pi)$ .

When Gaussian noise is passed through a narrow bandpass filter, the output can be written as

$$n(t) = X(t) \cos(\omega t) + Y(t) \sin(\omega t), \quad (2)$$

where  $X(t)$  and  $Y(t)$  are both independent and identically distributed Gaussian random variables, with zero mean and variance  $\sigma^2$ . By combining both (1) and (2), the radar signal return, in additive Gaussian noise, at the detector is

$$\begin{aligned} \zeta(t) &= s(t) + n(t) \\ &= (A \cos \theta + X(t)) \cos(\omega t) + (A \sin \theta + Y(t)) \sin(\omega t) \\ &= (a + X(t)) \cos(\omega t) + (b + Y(t)) \sin(\omega t) \\ &= R(t) \cos(\omega t - \Phi(t)), \end{aligned} \quad (3)$$

where

$$R(t) = \sqrt{(a + X(t))^2 + (b + Y(t))^2} \text{ and } \Phi(t) = \arctan \left[ \frac{b + Y(t)}{a + X(t)} \right]. \quad (4)$$

It can be shown that the joint density of  $(R, \Phi)$  is

$$f_{(R, \Phi)}(r, \phi | \theta) = \frac{r}{2\pi\sigma^2} \exp \left[ -\frac{r^2 + a^2 + b^2 - 2ra \cos \phi - 2rb \sin \phi}{2\sigma^2} \right], \quad (5)$$

where  $\sigma$  is as defined previously. To obtain the marginal probability density function (pdf) of the amplitude  $R(t)$ , we integrate the density (5) over all phases to obtain

$$\begin{aligned} f_R(r | \theta) &= \int_0^{2\pi} f_{(R, \Phi)}(r, \phi | \theta) d\phi \\ &= \frac{r}{\sigma^2} \exp \left[ -\frac{r^2 + A^2}{2\sigma^2} \right] I_0 \left[ \frac{rA}{\sigma^2} \right], \end{aligned} \quad (6)$$

where  $I_0$ , the modified Bessel function, of the first kind, of order zero, is defined to be

$$I_0(x) = \int_0^{2\pi} \exp(x \cos(\theta - \psi)) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \cos \theta} d\theta. \quad (7)$$

Note that (6) does not depend on  $\theta$ , so that  $f_R(r | \theta) = f_R(r)$ . We are now in a position where we can specify the probabilities of false alarm and detection for a single pulse. The classical approach to radar detection is to declare a target present when some statistic of the returned signal exceeds some pre-defined threshold level. A common choice in the literature is for the amplitude ( $R(t)$ ) to exceed a mean level [see Levanon 1988]. We suppose this threshold is  $\mu_\tau$ . Observe that, in view of (6), in the case where there is no signal present in the return, so that  $A = 0$ , the pdf of the amplitude is Rayleigh distributed with parameter  $\sigma$ . The probability of false alarm is the probability of declaring a target present when there is only noise. Hence, under the Neyman-Pearson criterion,

$$\mathbb{P}_{FA} = \int_{\mu_\tau}^{\infty} \frac{r}{\sigma^2} \exp \left[ -\frac{r^2}{2\sigma^2} \right] dr = \exp \left[ -\frac{\mu_\tau^2}{2\sigma^2} \right]. \quad (8)$$

The probability of detection is the probability of correctly declaring a target present. Under the Neyman-Pearson Criterion, for the case of a band-limited signal (1) in bandlimited Gaussian noise, it is given by the integral

$$\mathbb{P}_D = \int_{\mu_\tau}^{\infty} \frac{r}{\sigma^2} \exp \left[ -\frac{r^2 + A^2}{2\sigma^2} \right] I_0 \left[ \frac{rA}{\sigma^2} \right] dr, \quad (9)$$

where the SNR is given by  $\varsigma = \frac{A^2}{2\sigma^2}$  [see Levanon 1988]. The larger the signal to noise strength, the more likely we are to detect the target. In order to emphasize the dependence of (9) on the SNR  $\varsigma$  and the threshold  $\tau$ , we will write the detection probability as  $\rho(\varsigma, \tau)$ . By a change of variables  $r = \sqrt{2\sigma^2\nu}$ , and using the definition of  $\varsigma$ , it can be shown that (9) is equivalent to

$$\mathbb{P}_D = \int_{\tau}^{\infty} e^{-(\nu+\varsigma)} I_0(2\sqrt{\nu\varsigma}) d\nu := \rho(\varsigma, \tau), \quad (10)$$

where  $\tau = \frac{\mu_\tau^2}{2\sigma^2}$ . The detection probability (10) is the classical expression that can be found in [Levanon 1988] and [Minkler and Minkler 1990].

### 1.3 The Marcum Q-Function

The Marcum Q-Function arises in communications and radar signal processing problems. It has a long history in the study of target detection by pulsed radars [Marcum 1950, Marcum 1960 and Marcum and Swerling 1960]. As pointed out in [Simon 1998 and Simon and Alouini 2003], it occurs in performance analysis related to partially coherent, differentially coherent and noncoherent communications. Performance measures where the Q-Function arise are error probabilities in transmission over fading channels, and detection probability for code acquisition in direct sequence code division multiple access systems [Corazza and Ferrari 2002]. A specific example can be found in [Chiani 1999], who applies the Q-Function to performance evaluation of noncoherent and differentially coherent detection of digital modulation over Nakagami fading channels. In radar signal processing, the Q-Function appears as a detection probability. In particular, it is related to the probability of detection on  $n$  incoherently integrated received signals, in a Gaussian clutter and noise environment [Helstrom 1968, Nuttall 1975 and Shnidman 1989].

The standard Marcum Q-Function is defined by the integral

$$Q(\alpha, \beta) := \int_{\beta}^{\infty} x e^{-\left(\frac{x^2 + \alpha^2}{2}\right)} I_0(\alpha x) dx. \quad (11)$$

To see the connection (11) has to (10), introduce the transformation  $\alpha = \sqrt{2\varsigma}$  and  $\beta = \sqrt{2\tau}$ . Then by introducing the transformation  $\nu = \frac{x^2}{2}$  in integral (11), we obtain

$$\begin{aligned} Q(\alpha, \beta) &= Q(\sqrt{2\varsigma}, \sqrt{2\tau}) \\ &= e^{-\varsigma} \int_{\sqrt{2\tau}}^{\infty} x e^{-\frac{x^2}{2}} I_0(\sqrt{2\varsigma}x) dx \\ &= e^{-\varsigma} \int_{\tau}^{\infty} e^{-\nu} I_0(2\sqrt{\nu\varsigma}) d\nu. \end{aligned} \quad (12)$$

Hence it follows that the single pulse probability of detection  $\rho(\varsigma, \tau) = Q(\sqrt{2\varsigma}, \sqrt{2\tau})$ .

One of the issues associated with (11), and in particular, the generalised Marcum Q-Function to be introduced in Section 2, is how to estimate it well. Also of interest is obtaining upper and lower bounds on it. A number of authors have investigated these problems, including [Chiani 1999, Corazza and Ferrari 2002, Simon 1998, Simon and Alouini 2003 and Shnidman 1989].

## 1.4 Contributions of this Report

This report outlines new mathematical research into the detection probability (10), and its generalisation to  $n$  incoherently integrated pulses in Gaussian clutter. Specifically, [Shnidman 1989] does not identify a probabilistic interpretation to this detection probability, which can be used as a mechanism for generating a Monte Carlo estimator for the generalised Marcum Q-Function. A new interpretation of the detection probability, in Gaussian clutter and noise, of a number of incoherently integrated received signals is explored. In particular, it will be shown that this detection probability is equivalent to a probability associated with two independent Poisson variables. This generalises the case employed in [Weinberg and Kyprianou 2005].

In addition, some new interesting properties of this detection probability are explored mathematically. It is shown that the generalised detection probability under investigation is the solution to two stochastic Volterra integral equation of the second kind [Davis 1962 and Tricomi 1957]. This leads to useful representations that facilitate the construction of bounds on the detection probability, and consequently, the Marcum Q-Function. We will investigate the performance of some new bounds on the single pulse probability of detection (10).



## 2 Marcum's Q-Function and a Poisson Connection

### 2.1 Generalised Marcum Q-Function

The generalised Marcum Q-Function of order  $n \in \mathbb{N}$  [Corazza and Ferrari 2002, Nuttall 1975, Simon 1998 and Simon and Alouini 2003] is defined by the integral

$$Q_n(\alpha, \beta) = \frac{1}{\alpha^{n-1}} \int_{\beta}^{\infty} x^n e^{-\left(\frac{x^2 + \alpha^2}{2}\right)} I_{n-1}(\alpha x) dx, \quad (13)$$

where  $I_k(x)$  is the modified Bessel function, of the first kind, of order  $k$  [Bowman 1958], which can be defined by the integral formula

$$I_k(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (-ie^{-i\theta})^k e^{-x \sin \theta} d\theta = \frac{1}{\pi} \int_0^{\pi} e^{x \cos \theta} \cos(k\theta) d\theta, \quad (14)$$

where  $i^2 = -1$ . We will be exploring this function throughout this report. In particular, we are interested in it because of its connection to radar detection probabilities. As pointed out in [Shnidman 1989], the detection probability of  $n$  incoherently integrated received signals in a Gaussian noise and clutter environment is given by

$$\rho_n(\varsigma, \tau) = \int_{\tau}^{\infty} \left(\frac{\nu}{n\varsigma}\right)^{\frac{(n-1)}{2}} e^{-(\nu+n\varsigma)} I_{n-1}(2\sqrt{\nu n\varsigma}) d\nu, \quad (15)$$

when using a quadratic detector via the Neyman-Pearson Criterion.

In [Weinberg and Kyprianou 2005], (15) was investigated in the case where  $n = 1$ . It was shown that (15) for  $n = 1$  is identical to the probability that one Poisson random variable, representing a threshold variable, is less than or equal to another Poisson, representing the SNR. We will show that this result can be generalised to (15).

It can be shown [Helstrom 1968, Nuttall 1975 and Shnidman 1989] that (15) is related to (13) through

$$\rho_n(\varsigma, \tau) = Q_n(\sqrt{2n\varsigma}, \sqrt{2\tau}). \quad (16)$$

Hence, any new expression obtained for either one of (13) and (15) can be applied to the other, through (16).

### 2.2 The Poisson Connection

We now derive a new probabilistic expression for (15). The following mathematics is included for completeness, and is based upon the analysis in [Shnidman 1989]. Recall that

the Bessel function of the first kind of order  $n \in \mathbb{N}$ , denoted  $J_n(x)$ , has a series expansion in terms of Gamma functions

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k! \Gamma(n+k+1)}, \quad (17)$$

[see Bowman 1958 and Tsyarkin and Tsyarkin 1988] and the modified Bessel function  $I_n(x)$  of the first kind of order  $n \in \mathbb{N}$  is related to  $J_n(x)$  through  $I_n(x) = (-i)^n J_n(ix)$ . Hence, applying this to (17), and using the fact that we will only be considering integral  $n$ , it is not difficult to construct the power series

$$I_n(x) = \sum_{k=0}^{\infty} \frac{x^{n+2k}}{2^{n+2k} k! (n+k)!}. \quad (18)$$

Hence, applying (18), to  $I_{n-1}(2\sqrt{\nu n \varsigma})$ , the detection probability (15) becomes

$$\begin{aligned} \rho_n(\varsigma, \tau) &= \frac{e^{-n\varsigma}}{(n\varsigma)^{\frac{n-1}{2}}} \int_{\tau}^{\infty} \nu^{\frac{n-1}{2}} e^{-\nu} \sum_{k=0}^{\infty} \frac{(\sqrt{\nu n \varsigma})^{n-1+2k}}{k! (n-1+k)!} d\nu \\ &= e^{-n\varsigma} \sum_{k=0}^{\infty} \frac{(n\varsigma)^k}{k! (n-1+k)!} \int_{\tau}^{\infty} e^{-\nu} \nu^{n-1+k} d\nu. \end{aligned} \quad (19)$$

It can be shown that

$$\int_{\tau}^{\infty} e^{-\nu} \nu^{n-1+k} d\nu = e^{-\tau} \int_0^{\infty} e^{-\nu} (\nu + \tau)^{n-1+k} d\nu. \quad (20)$$

An application of a binomial expansion to  $(\nu + \tau)^{n-1+k}$ , transforms (20) to

$$\int_0^{\infty} (\nu + \tau)^{n-1+k} e^{-\nu} d\nu = \sum_{j=0}^{n-1+k} \frac{(n-1+k)! \tau^j}{j!}. \quad (21)$$

Hence, applying (21) to (19), we arrive at the double series expansion

$$\rho_n(\varsigma, \tau) = e^{-(\tau+n\varsigma)} \sum_{k=0}^{\infty} \frac{(n\varsigma)^k}{k!} \sum_{j=0}^{n-1+k} \frac{\tau^j}{j!}. \quad (22)$$

The expansion (22) is not new to radar, but can be found in a number of references, such as [Shnidman 1989]. The following interpretation has not been identified previously. Define a pair of independent random variables  $(\aleph_1(n\varsigma), \aleph_2(\tau)) \stackrel{d}{=} (\mathbf{Po}(n\varsigma), \mathbf{Po}(\tau))$ . We observe that (22) can be written in the form

$$\begin{aligned} \rho_n(\varsigma, \tau) &= \sum_{k=0}^{\infty} \frac{e^{-n\varsigma} (n\varsigma)^k}{k!} \sum_{j=0}^{n-1+k} \frac{e^{-\tau} \tau^j}{j!} \\ &= \sum_{k=0}^{\infty} \mathbb{P}[\aleph_1(n\varsigma) = k] \mathbb{P}[\aleph_2(\tau) \leq n-1+k] \\ &= \mathbb{P}(\aleph_2(\tau) \leq n-1 + \aleph_1(n\varsigma)). \end{aligned} \quad (23)$$

This result complements the corresponding result for  $n = 1$  in [Weinberg and Kyprianou 2005].

From a radar analysis point of view, (23) is an unexpected result. We can explain this result in a heuristic manner as follows. We can think of the random variable  $n - 1 + \aleph_1(n\varsigma)$  as representing the signal to noise ratio. Similarly,  $\aleph_2(\tau)$  is a random variable representing the detection threshold. If we condition on the threshold variable, then (23) is counting the number of exceedences of a randomised threshold level. The larger the SNR  $\varsigma$ , the more exceedences of the threshold, since  $\varsigma$  is also the mean value of the associated Poisson distribution. Also, Poisson variables are models for “rare” events, and detections, in some circumstances such as low SNR, can be thought of as such. Hence the form of (23) is intuitive because it is counting the number of exceedences of a randomised threshold, using a model for rare events.

We can also apply (23) to the generalised Marcum Q-Function. Note that, with reference to (16), (23) implies that

$$Q_n(\alpha, \beta) = \mathbb{P} \left( \aleph_2 \left( \frac{\beta^2}{2} \right) \leq n - 1 + \aleph_1 \left( \frac{\alpha^2}{2} \right) \right). \quad (24)$$

Both (23) and (24) lead to very simple Monte Carlo estimators for these respective functions.

### 3 Stochastic Representation

This Section is concerned with the derivation of a further stochastic representation of the detection probability (15). In order to derive this result, we require an expression for the distributional differences between independent Poisson random variables. Using Stein's method, a new integral expression for this is obtained, and is applied to (23).

#### 3.1 Stein's Method for Poisson Approximation

Stein's method [Barbour *et. al.* 1992, Chen 1975 and Stein 1972] is a general scheme that enables one to obtain estimates of the rate of convergence of probability distributions. [Weinberg 2005] contains a detailed description of the method, its history and application. Our application of it is to construct an expression measuring the distributional difference between two independent Poisson random variables. The application of such an expression is to the construction of further stochastic representations of (23). In this Subsection we present a concise outline of Stein's method for Poisson approximation. The reader is advised to consult [Weinberg 2005] for a more comprehensive discussion.

The key idea behind Stein's method for Poisson approximation is to find a solution to the Stein equation

$$\lambda g(j+1) - jg(j) = f(j) - \mathbf{Po}(\lambda)f, \quad (25)$$

where  $f(j) = \mathbb{I}_{[j \in A]}$ ,  $\lambda > 0$  and

$$\mathbb{I}_{[j \in A]} = \begin{cases} 1 & \text{if } j \in A; \\ 0 & \text{otherwise.} \end{cases}$$

If (25) is well defined, then for a random variable  $W$  with support the nonnegative integers,

$$\mathbb{E}[\lambda g(W+1) - Wg(W)] = \mathbb{P}(W \in A) - \mathbf{Po}(\lambda)\{A\}. \quad (26)$$

Hence, to measure how well the distribution of  $W$  is approximated by that of a Poisson random variable, we need to bound or estimate the left hand side of (26).

The probabilistic approach to Stein's method [Barbour *et. al.* 1992] relates (25) to the generator of an immigration-death process, with immigration rate  $\lambda$ , and unit per capita death rate [Ross 1983]. The generator of such a Markov process is

$$\mathcal{A}h(j) = \lambda[h(j+1) - h(j)] - j[h(j) - h(j-1)]. \quad (27)$$

Here, we have written  $g(j+1) = h(j+1) - h(j)$ . The probabilistic form of the Stein equation (25) is

$$\mathcal{A}h(j) = f(j) - \mathbf{Po}(\lambda)f, \quad (28)$$

and has solution

$$h(j) = - \int_0^\infty \mathbb{E}[f(Z_j(t)) - \mathbf{Po}(\lambda)f]dt, \quad (29)$$

where  $Z_j(t)$  is the immigration-death process, starting with  $j$  individuals [see Weinberg 2005]. The idea, in this context, is to estimate the distributional differences in (28) by estimating the expectation of the generator (27), using properties of the function (29) and Markov process theory.

In the next Subsection, we apply this to estimate the difference in distribution of two independent Poisson random variables.

### 3.2 Differences of Poisson Distributions

We require an expression for the difference in distribution of  $W \stackrel{d}{=} \mathbf{Po}(\mu)$  and  $V \stackrel{d}{=} \mathbf{Po}(\lambda)$ . Hence, we simplify (26). Note that

$$\begin{aligned} \mathbb{E}[Wg(W)] &= \sum_{j=0}^{\infty} \frac{jg(j)e^{-\mu}\mu^j}{j!} \\ &= \mu \sum_{j=1}^{\infty} \frac{g(j)e^{-\mu}\mu^{j-1}}{(j-1)!} \\ &= \mu \mathbb{E}[g(W+1)]. \end{aligned} \tag{30}$$

Hence, applying (30) to (26), it follows that

$$\mathbb{E}[\lambda g(W+1) - Wg(W)] = (\lambda - \mu) \mathbb{E}[g(W+1)]. \tag{31}$$

Consequently, we need to estimate the expectation  $\mathbb{E}[g(W+1)]$ . It is useful to now use the probabilistic interpretation to Stein's method, in order to simplify (31). Hence, by applying (29), and writing  $g(j+1) = h(j+1) - h(j)$ ,

$$g(j+1) = - \int_0^{\infty} \mathbb{E}[f(Z_{j+1}(t)) - f(Z_j(t))] dt. \tag{32}$$

As in [Barbour *et. al.* 1992 and Lindvall 1992], we couple the two processes in (32) as follows. The process  $Z_{j+1}(t)$  evolves as  $Z_j(t)$  with the addition of an extra individual, with independent Exponentially distributed lifetime. Hence  $Z_{j+1}(t) = Z_j(t) + \mathbb{I}_{[\zeta > t]}$ , where  $\zeta$  is an independent Exponential random variable with mean 1. Then after  $t \geq \zeta$ , the processes couple together. It thus follows that

$$\begin{aligned} g(j+1) &= -\mathbb{E} \int_0^{\infty} \mathbb{I}_{[\zeta > t]} [f(Z_j(t) + 1) - f(Z_j(t))] dt \\ &= - \int_0^{\infty} e^{-t} \mathbb{E}[f(Z_j(t) + 1) - f(Z_j(t))] dt. \end{aligned} \tag{33}$$

In view of (31) and (33), what we now require is the distribution of the immigration-death process  $Z_j(t)$  when  $j$  is a Poisson random variable with mean  $\mu$ . The following Lemma gives the required result:

**Lemma 3.1** *For the immigration-death process  $Z_j(t)$ , if  $W \stackrel{d}{=} \mathbf{Po}(\mu)$ , then  $Z_W(t) \stackrel{d}{=} \mathbf{Po}(\lambda + (\mu - \lambda)e^{-t})$ .*

Proof:

Let  $P_X(s) = \mathbb{E}[s^X]$  be the probability generating function of the random variable  $X$ . It can be shown that if  $X \stackrel{d}{=} \mathbf{Po}(\nu)$ , then  $P_X(s) = e^{-\nu(1-s)}$ . Also, if  $Y \stackrel{d}{=} \mathbf{Bin}(n, p)$ , then  $P_Y(s) = (1-p+ps)^n$ . The immigration-death process  $Z_j(t)$  can be decomposed into a sum of two independent random variables. Namely, it can be shown that  $Z_j(t) = X_j(t) + Z_0(t)$ , where  $X_j(t) \stackrel{d}{=} \mathbf{Bin}(j, e^{-t})$  and  $Z_0(t) \stackrel{d}{=} \mathbf{Po}(\lambda(1 - e^{-t}))$  [see Barbour *et. al.*, 1992]. By using the independence of these two variables, and by conditioning on  $W$ , it follows that

$$\begin{aligned}
P_{Z_W(t)} &= \mathbb{E}[s^{Z_W(t)}] = \mathbb{E}[\mathbb{E}[s^{Z_W(t)}|W]] \\
&= \mathbb{E}[\mathbb{E}[s^{X_W(t)}|W]\mathbb{E}[s^{Z_0(t)}|W]] \\
&= \mathbb{E}[P_{X_W(t)}(s)]P_{Z_0(t)}(s) \\
&= \mathbb{E}[(1 - e^{-t} + e^{-t}s)^W]e^{-\lambda(1-e^{-t})(1-s)} \\
&= e^{-[\lambda+(\mu-\lambda)e^{-t}](1-s)},
\end{aligned}$$

implying the desired result.  $\square$

Since we are interested in cumulative distribution functions, we make the choice of  $A = \{0, 1, \dots, k\}$ , for some  $k \in \mathbb{N}$ . This is equivalent to the choice of  $f(j) = \mathbb{I}[j \leq k]$ . Let  $X(t) = Z_W(t)$ . Then it follows that

$$\begin{aligned}
\mathbb{E}[g(W+1)] &= - \int_0^\infty e^{-t} [\mathbb{P}(X(t) + 1 \leq k) - \mathbb{P}(X(t) \leq k)] dt \\
&= \int_0^\infty e^{-t} \mathbb{P}(X(t) = k) dt.
\end{aligned} \tag{34}$$

Lemma 3.1 implies the point probabilities of  $X(t)$  are given by

$$\mathbb{P}(X(t) = k) = e^{[-\lambda+(\mu-\lambda)e^{-t}]} \frac{[\lambda + (\mu - \lambda)e^{-t}]^k}{k!}. \tag{35}$$

Applying (35) to (34), and making the transformation  $\nu = \lambda + (\mu - \lambda)e^{-t}$ , and without loss of generality, considering  $\lambda \geq \mu$ , we obtain

$$\mathbb{E}[g(W+1)] = \frac{1}{\lambda - \mu} \int_\mu^\lambda \frac{e^{-\nu} \nu^k}{k!} d\nu. \tag{36}$$

An application of (36) to (31) results in

$$\mathbf{Po}(\mu)f - \mathbf{Po}(\lambda)f = \int_\mu^\lambda \frac{e^{-\nu} \nu^k}{k!} d\nu. \tag{37}$$

Hence (37) expresses the differences in Poisson cumulative probability distributions as an integral over their respective means, of a Poisson distribution function whose mean is the integration variable. The following subsection will apply (37) to construct an alternative expression for (23).

Note that if we introduce a random variable  $\Theta(\mu, \lambda) \stackrel{d}{=} \mathbf{R}(\mu, \lambda)$ , and an independent Poisson random variable  $\aleph(\nu) \stackrel{d}{=} \mathbf{Po}(\nu)$ , then it follows that (37) is equivalent to

$$\mathbf{Po}(\mu)f - \mathbf{Po}(\lambda)f = (\lambda - \mu)\mathbb{P}[\aleph(\Theta(\lambda, \mu)) = k]. \quad (38)$$

The random variable  $\aleph(\Theta(\lambda, \mu))$  in (38), a Poisson with a random mean, is called a mixed Poisson distribution.

### 3.3 Stochastic Representation I

We are now in a position to derive new expressions for the detection probability (23). It is necessary to introduce a number of random variables. Let  $\aleph_1(n\varsigma) \stackrel{d}{=} \mathbf{Po}(n\varsigma)$  and  $\aleph_2(\tau) \stackrel{d}{=} \mathbf{Po}(\tau)$  be the two independent Poisson random variables in the representation (23). Introduce random variables  $\aleph_3(\varsigma) \stackrel{d}{=} \mathbf{Po}(\varsigma)$ ,  $\aleph_4(\nu) \stackrel{d}{=} \mathbf{Po}(\nu)$  and  $\Theta(\tau, \varsigma) \stackrel{d}{=} \mathbf{R}(\tau \wedge \varsigma, \tau \vee \varsigma)$ . We assume that both  $\aleph_1(n\varsigma)$  and  $\aleph_3(\varsigma)$  are independent, and when  $n = 1$ , it is assumed they are independent copies of each other [Lindvall 1992]. Additionally, it is assumed that  $\aleph_1(n\varsigma)$ ,  $\aleph_4(\nu)$  and  $\Theta(\tau, \varsigma)$  are pairwise independent.

Then by conditioning on  $\aleph_1(n\varsigma)$ ,

$$\begin{aligned} \rho_n(\varsigma, \tau) &= \mathbb{P}[\aleph_2(\tau) \leq n - 1 + \aleph_1(n\varsigma)] \\ &= \sum_{k=0}^{\infty} \mathbb{P}[\aleph_1(n\varsigma) = k] \mathbb{P}[\aleph_2(\tau) \leq n - 1 + k] \\ &= \sum_{k=0}^{\infty} \mathbb{P}[\aleph_1(n\varsigma) = k] \left( \mathbb{P}[\aleph_2(\tau) \leq n - 1 + k] - \mathbb{P}[\aleph_3(\varsigma) \leq n - 1 + k] \right) \\ &\quad + \sum_{k=0}^{\infty} \mathbb{P}[\aleph_1(n\varsigma) = k] \mathbb{P}[\aleph_3(\varsigma) \leq n - 1 + k]. \end{aligned} \quad (39)$$

Since, by construction,  $\aleph_1(n\varsigma)$  and  $\aleph_3(\varsigma)$  are independent and Poisson distributed, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \mathbb{P}[\aleph_1(n\varsigma) = k] \mathbb{P}[\aleph_3(\varsigma) \leq n - 1 + k] &= \mathbb{P}[\aleph_3(\varsigma) \leq n - 1 + \aleph_1(n\varsigma)] \\ &= \rho_n(\varsigma, \varsigma), \end{aligned} \quad (40)$$

where we have utilised the Poisson expression (23) for the detection probability (15). Also, with an application of the Poisson difference equation (37), we have

$$\mathbb{P}[\aleph_2(\tau) \leq n - 1 + k] - \mathbb{P}[\aleph_3(\varsigma) \leq n - 1 + k]$$

$$\begin{aligned}
&= \int_{\tau}^{\varsigma} \frac{e^{-\nu} \nu^{n-1+k}}{k!} d\nu \\
&= \int_{\tau}^{\varsigma} \mathbb{P}[\aleph_4(\nu) = n-1+k] d\nu.
\end{aligned} \tag{41}$$

Consequently, by applying (41) and using the fact that  $\aleph_1(n\varsigma)$  and  $\aleph_4(\nu)$  are independent,

$$\begin{aligned}
\sum_{k=0}^{\infty} \mathbb{P}[\aleph_1(n\varsigma) = k] &= \left( \mathbb{P}[\aleph_2(\tau) \leq n-1+k] - \mathbb{P}[\aleph_3(\varsigma) \leq n-1+k] \right) \\
&= \int_{\tau}^{\varsigma} \mathbb{P}[\aleph_4(\nu) = n-1+\aleph_1(n\varsigma)] d\nu.
\end{aligned} \tag{42}$$

Finally, combining (40) and (42), (39) becomes

$$\rho_n(\varsigma, \tau) = \rho_n(\varsigma, \varsigma) + \int_{\tau}^{\varsigma} \mathbb{P}[\aleph_4(\nu) = n-1+\aleph_1(n\varsigma)] d\nu. \tag{43}$$

This gives a new expression for the detection probability (15). It shows that the latter is equal to the same detection probability, where the threshold is equal to the SNR, plus or minus a discrepancy factor that depends on the difference between  $\tau$  and  $\varsigma$ .

It is interesting to note that the integrand in (43) is closely related to the detection probability  $\rho_n(\varsigma, \nu) = \mathbb{P}[\aleph_4(\nu) \leq n-1+\aleph_1(n\varsigma)]$ . In fact, it is obvious that  $\mathbb{P}[\aleph_4(\nu) = n-1+\aleph_1(n\varsigma)] = \rho_n(\varsigma, \nu) - \mathbb{P}[\aleph_4(\nu) < n-1+\aleph_1(n\varsigma)]$ . Hence, the integrand is a nonlinear function of  $\rho_n(\varsigma, \nu)$ . This suggests that the representation (43) is a nonlinear integral equation of  $\rho_n(\varsigma, \nu)$ , as a function of its second independent variable, namely  $\nu$ .

As pointed out in [Davis 1962 and Tricomi 1957], a function  $f(x)$ , defined on a suitable domain, is the solution to a nonlinear Volterra integral equation if it satisfies

$$f(x) = f(x_0) + \int_{x_0}^x F[t, f(t)] dt, \tag{44}$$

where  $F$  is nonlinear. In the case of Volterra integral equations of the first and second kind, it is assumed that the function  $F$  is a product of the function  $f$  and a kernel  $K$ . We illustrate how (43) is of the form (44).

Write the integrand in (43) as

$$\mathbb{P}[\aleph_4(\nu) = n-1+\aleph_1(n\varsigma)] = \rho_n(\varsigma, \nu) K_n(\varsigma, \nu), \tag{45}$$

where the kernel  $K_n(\varsigma, \nu)$  is

$$K_n(\varsigma, \nu) = \frac{\mathbb{P}[\aleph_4(\nu) = n-1+\aleph_1(n\varsigma)]}{\mathbb{P}[\aleph_4(\nu) \leq n-1+\aleph_1(n\varsigma)]}. \tag{46}$$

Consider  $\varsigma$  (and  $n$ ) fixed, and choose  $f(x) = \rho_n(\varsigma, x)$ ,  $F[x, f(x)] = -f(x)K_n(\varsigma, x)$  and  $x_0 = \varsigma$ . Then (43) is exactly of the form (44), as a function of the threshold, namely  $x = \tau$ .



It is worth noting that a function  $f$  satisfying the integral equation (44) can be estimated through a Picard iteration scheme [Giles 1987]. One considers the functional series  $\{f_m, m \in \mathbb{N}\}$  with  $f_0(x) = f(x_0)$  and  $f_m(x)$  defined recursively through

$$f_m(x) = f(x_0) + \int_{x_0}^x F[t, f_{m-1}(t)] dt. \quad (47)$$

There are existence and uniqueness theorems, which give conditions on  $F$  to guarantee a reasonable approximation is obtained [Giles 1987 and Tricomi 1957]. In principle, one could apply the scheme (47) to (43) to obtain estimates of the detection probability  $\rho_n(\varsigma, \tau)$ . In addition, it may be possible to use properties of the solution to Volterra integral equations to derive new bounds on the Marcum Q-function. The validity and merit of these approaches will be investigated in future research.

Observe that, by taking partial derivatives, with respect to  $\tau$ , it is not difficult to see that

$$\frac{\partial}{\partial \tau} \rho_n(\varsigma, \tau) = -\mathbb{P}[\aleph_4(\tau) = n - 1 + \aleph_1(n\varsigma)], \quad (48)$$

implying that  $\rho_n(\varsigma, \tau)$  is a decreasing function of the threshold  $\tau$ , for fixed  $\varsigma$ . Note that, in view of the preceding comments, the partial derivative (48) is a nonlinear function of  $\rho_n(\varsigma, \tau)$ .

It is still possible to write (43) in a more compact stochastic form. Using the definition of  $\Theta(\tau, \varsigma)$ , it is not difficult to see that

$$\rho_n(\varsigma, \tau) = \rho_n(\varsigma, \varsigma) + (\varsigma - \tau) \mathbb{P}[\aleph_4(\Theta(\tau, \varsigma)) = n - 1 + \aleph_1(n\varsigma)]. \quad (49)$$

Both (43) and (49) are new expressions for the detection probability (15), and both can be applied to the generalised Marcum Q-Function (13), by an application of (16), to derive analogous expressions.

### 3.4 Stochastic Representation II

It is possible to derive a representation analogous to (43), by instead conditioning on  $\aleph_2(\tau)$ , instead of conditioning on  $\aleph_1(n\varsigma)$ . Introduce a random variable  $\aleph_5(n\tau) \stackrel{d}{=} \mathbf{Po}(n\tau)$ , which we assume to be independent of  $\aleph_2(\tau)$ . Throughout we use the same definitions of appropriate random variables, as defined in Subsection 3.3. Then, using a similar argument as in the derivation of (43),

$$\begin{aligned} \rho_n(\varsigma, \tau) &= \mathbb{P}[\aleph_2(\tau) \leq n - 1 + \aleph_1(n\varsigma)] \\ &= \sum_{k=0}^{\infty} \mathbb{P}[\aleph_2(\tau) = k] (\mathbb{P}[\aleph_1(n\varsigma) \geq k - n + 1] - \mathbb{P}[\aleph_5(n\tau) \geq k - n + 1]) \\ &\quad + \sum_{k=0}^{\infty} \mathbb{P}[\aleph_2(\tau) = k] \mathbb{P}[\aleph_5(n\tau) \geq k - n + 1] \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \mathbb{P}[\aleph_2(\tau) = k] (\mathbb{P}[\aleph_5(n\tau) \leq k - n] - \mathbb{P}[\aleph_1(n\varsigma) \leq k - n]) \\
&\quad + \mathbb{P}[\aleph_2(\tau) \leq \aleph_5(n\tau) + n - 1] \\
&= \rho_n(\tau, \tau) + \int_{n\tau}^{n\varsigma} \mathbb{P}[\aleph_4(\nu) = \aleph_2(\tau) - n] d\nu \\
&\equiv \rho_n(\tau, \tau) + \int_{\tau}^{\varsigma} n \mathbb{P}[\aleph_4(n\nu) = \aleph_2(\tau) - n] d\nu. \tag{50}
\end{aligned}$$

Note that, analogously to (48), by taking partial derivatives of (50) with respect to  $\varsigma$ ,

$$\frac{\partial}{\partial \varsigma} \rho_n(\varsigma, \tau) = n \mathbb{P}[\aleph_4(n\varsigma) = \aleph_2(\tau) - n]. \tag{51}$$

Consequently, (51) shows that  $\rho_n(\varsigma, \tau)$  is an increasing function of the SNR parameter  $\varsigma$ , for fixed  $\tau$ .

We can also construct an analogue of (49). Using the same definitions as in the latter expression,

$$\rho_n(\varsigma, \tau) = \rho_n(\tau, \tau) + n(\varsigma - \tau) \mathbb{P}[\aleph_4(n\Theta(\tau, \varsigma)) = \aleph_2(\tau) - n]. \tag{52}$$

The representations (43) and (50) indicate that the detection probability (15), and also the Marcum Q-Function (13), may be estimated using properties of Volterra integral equations. This idea is partially explored in the next Section.

## 4 Bounds on the Pulse Detection Probability

In this final Section we consider bounding the detection probability (15) in the simple case of  $n = 1$ . We will refer to this case as the *pulse detection probability*. Bounding this case, and equivalently, obtaining bounds on the standard Marcum Q-Function (11), has been the subject of much interest in recent years [Chiani 1999, Corazza and Ferrari 2002, Simon 1998 and Simon and Alouini (2000 and 2003)]. We investigate whether bounds derived from the new stochastic representations (43) and (50) can improve known bounds on this pulse detection probability.

It is important to note that a closed form result exists for the pulse detection probability, in the case where  $\varsigma = \tau$ . Specifically, it can be shown [Schwartz, Bennett and Stein 1966] that

$$\rho(\varsigma, \varsigma) = \frac{1}{2} \left[ 1 + e^{-2\varsigma} I_0(2\varsigma) \right]. \quad (53)$$

This result will be an integral component of the new bounds. A probabilistic proof of (53) can be found in Appendix A. Before deriving some new bounds based on (43) and (50), we examine some well-known bounds.

### 4.1 Lower and Upper Bounds

To begin, we examine known lower and upper bounds of the detection probability (15) in the case of  $n = 1$ . Such bounds have been examined for the Marcum Q-Function (13), for  $n = 1$ , in [Chiani 1999, Corazza and Ferrari 2002, Simon 1998 and Simon and Alouini (2000 and 2003)]. In the following we will express their bounds in terms of the single pulse probability of detection. Tight lower bounds give estimates of the least possible values of the detection probability, and thus have been used as a performance measure in [Chiani 1999]. Upper bounds indicate the maximum possible detection probability, and are also of interest in performance analysis. Unless otherwise stated, it will be understood that all lower bounds are taken, in practice, as the maximum of the expressed bound and zero, in order to avoid meaningless lower bounds. Similarly, all upper bounds are the minimum of the expressed bound and unity.

#### 4.1.1 Case 1: $\varsigma > \tau$

An upper bound on the pulse detection probability, or equivalently, the standard Marcum Q-function, for the case where  $\varsigma > \tau$  have only recently appeared in [Corazza and Ferrari 2002]. These authors derive the upper bound

$$\begin{aligned} \rho(\varsigma, \tau) \leq & 1 - \frac{I_0(2\sqrt{\varsigma\tau})}{e^{2\sqrt{\varsigma\tau}}} \left\{ e^{-\varsigma} - e^{-\frac{1}{2}[\sqrt{2\tau} - \sqrt{2\varsigma}]^2} \right. \\ & \left. + \sqrt{\pi\varsigma} [\mathbf{Erfc}(-\sqrt{\varsigma}) - \mathbf{Erfc}(\sqrt{\tau} - \sqrt{\varsigma})] \right\}, \end{aligned} \quad (54)$$

where **Erfc** is the complementary error function

$$\mathbf{Erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt. \quad (55)$$

As [Corazza and Ferrari 2002] emphasize, the upper bound (54) is the only known non-trivial upper bound on the pulse detection probability for the case under consideration.

Lower bounds for  $\varsigma > \tau$  are more prevalent in the literature. A lower bound analogous to (54) is also introduced in [Corazza and Ferrari 2002], which is

$$\begin{aligned} \rho(\varsigma, \tau) \geq & 1 - e^{-\frac{1}{2}[2\varsigma+\eta^2]} \left\{ e^{-\frac{\eta^2}{2}} - e^{-\frac{1}{2}[\sqrt{2\tau}-\eta]^2} \right. \\ & \left. + \eta \sqrt{\frac{\pi}{2}} \left[ \mathbf{Erfc}\left(-\frac{\eta}{\sqrt{2}}\right) - \mathbf{Erfc}\left(\frac{\sqrt{2\tau}-\eta}{\sqrt{2}}\right) \right] \right\}, \end{aligned} \quad (56)$$

where  $\eta = \frac{\log(I_0(2\sqrt{\varsigma\tau}))}{\sqrt{2\tau}}$ .

A Chernoff-like lower bound is derived in [Simon and Alouini 2000], given by

$$\rho(\varsigma, \tau) \geq 1 - \frac{1}{2} \left[ e^{-[\sqrt{\tau}-\sqrt{\varsigma}]^2} - e^{-[\sqrt{\tau}+\sqrt{\varsigma}]^2} \right]. \quad (57)$$

[Chiani 1999] also proposes a Chernoff-like lower bound, which is

$$\rho(\varsigma, \tau) \geq e^{-[\varsigma+\tau]} I_0(2\sqrt{\varsigma\tau}). \quad (58)$$

Finally, [Simon 1998] suggests the bound

$$\rho(\varsigma, \tau) \geq 1 - \frac{1}{\sqrt{\varsigma} - \sqrt{\tau}} e^{-[\varsigma-\tau]^2}. \quad (59)$$

We now include some comments on the performance and accuracy of these bounds. It is reported in [Corazza and Ferrari 2002] that the upper bound (54) performs well, when compared to the exact pulse detection probability, except as  $\tau$  increases. [Chiani 1999] states that the lower bound (57) is very tight and better than (58). [Corazza and Ferrari 2002] also report that their lower bound (56) is also very tight, and only matched by the bound (57). The bound (59) is reported to be quite unreliable by [Corazza and Ferrari 2002].

#### 4.1.2 Case 2: $\varsigma < \tau$

In this case, there are many proposed upper and lower bounds in the literature. [Corazza and Ferrari 2002] derive the two sided bound

$$\begin{aligned} \frac{I_0(2\sqrt{\varsigma\tau})}{e^{2\sqrt{\varsigma\tau}}} \sqrt{\pi\tau} \mathbf{Erfc}(\sqrt{\tau} - \sqrt{\varsigma}) & \leq \rho(\varsigma, \tau) \\ & \leq \frac{I_0(2\sqrt{\varsigma\tau})}{e^{2\sqrt{\varsigma\tau}}} \left\{ e^{-[\tau-\varsigma]^2} + \sqrt{\pi\varsigma} \mathbf{Erfc}(\sqrt{\tau} - \sqrt{\varsigma}) \right\}. \end{aligned} \quad (60)$$

[Simon and Alouini 2000] present the bound

$$e^{-[\tau+\varsigma]^2} \leq \rho(\varsigma, \tau) \leq e^{-[\tau-\varsigma]^2}. \quad (61)$$

[Chiani 1999] derives the two sided bound

$$e^{-[\varsigma+\tau]} I_0(2\sqrt{\varsigma\tau}) \leq \rho(\varsigma, \tau) \leq e^{-[\varsigma+\tau]} I_0(2\sqrt{\varsigma\tau}) + \frac{\sqrt{\varsigma\pi}}{2} \mathbf{Erfc}(\sqrt{\tau} - \sqrt{\varsigma}). \quad (62)$$

The final bound we consider is that due to [Simon 1998], which is

$$\frac{\sqrt{\tau}}{\sqrt{\tau} + \sqrt{\varsigma}} e^{-[\tau+\varsigma]^2} \leq \rho(\varsigma, \tau) \leq \frac{\sqrt{\tau}}{\sqrt{\tau} - \sqrt{\varsigma}} e^{-[\tau-\varsigma]^2}. \quad (63)$$

Again, we provide some comments on the accuracy and performance of these bounds. [Corazza and Ferrari 2002] compare their upper bound with the three others introduced here, and found that their upper bound in (60) has the best performance. Their lower bound in (60) also performs well, and is only matched by that in (62). [Chiani 1999] reports that the bounds in (62) are tighter than those in (61).

## 4.2 Some New Lower and Upper Bounds

In this section we derive some new upper and lower bounds on the detection probability (15) in the case where  $n = 1$ . These bounds are derived using the representations (43) and (50). We consider the two cases  $\varsigma > \tau$  and  $\varsigma < \tau$  separately. Observe that in view of (43) and (50), we have

$$\rho(\varsigma, \tau) = \rho(\varsigma, \varsigma) + \int_{\tau}^{\varsigma} \mathbb{P}[\aleph_1(\nu) = \aleph_2(\varsigma)] d\nu, \quad (64)$$

and

$$\rho(\varsigma, \tau) = \rho(\tau, \tau) + \int_{\tau}^{\varsigma} \mathbb{P}[\aleph_1(\nu) = \aleph_2(\tau) - 1] d\nu, \quad (65)$$

where  $\aleph_1(\cdot)$  and  $\aleph_2(\cdot)$  are independent Poisson random variables, with appropriate means. Throughout we will utilise the notation that  $a \vee b$  is the maximum of  $a$  and  $b$ , while  $a \wedge b$  is the minimum of  $a$  and  $b$ .

### 4.2.1 Case 1: $\varsigma > \tau$

Note that in this case both integrals in (64) and (65) are nonnegative, and so an appropriate lower bound is given by

$$\rho(\varsigma, \tau) \geq \rho(\varsigma, \varsigma) \vee \rho(\tau, \tau), \quad (66)$$

where we can apply (53) to evaluate this exactly.

In order to derive an upper bound, recall that (51) implies  $\rho(\varsigma, \tau)$  is an increasing function in  $\varsigma$ . Hence,

$$\begin{aligned}\rho(\varsigma, \tau) &\leq \rho(\varsigma, \varsigma) + \int_{\tau}^{\varsigma} \mathbb{P}[\aleph_2(\varsigma) \leq \aleph_1(\nu)] d\nu \\ &= \rho(\varsigma, \varsigma) + \int_{\tau}^{\varsigma} \rho(\nu, \varsigma) d\nu \\ &\leq [1 + \varsigma - \tau] \rho(\varsigma, \varsigma).\end{aligned}\tag{67}$$

By applying a similar argument to (65), and using the fact that in view of (48),  $\rho(\varsigma, \tau)$  is a decreasing function of  $\tau$ , it can be shown that

$$\rho(\varsigma, \tau) \leq [1 + \varsigma - \tau] \rho(\tau, \tau).\tag{68}$$

Hence, by combining (67) and (68), we arrive at the bound

$$\rho(\varsigma, \tau) \leq [1 + \varsigma - \tau] (\rho(\varsigma, \varsigma) \wedge \rho(\tau, \tau)).\tag{69}$$

Finally, by combining the bounds (66) and (69), we arrive at the two sided bound

$$(\rho(\varsigma, \varsigma) \vee \rho(\tau, \tau)) \leq \rho(\varsigma, \tau) \leq [1 + \varsigma - \tau] (\rho(\varsigma, \varsigma) \wedge \rho(\tau, \tau)).\tag{70}$$

On inspection of the upper bound (69) it is clear that if the difference  $\varsigma - \tau$  is much larger than minimum of the two probabilities, then the bound will be likely to exceed unity.

#### 4.2.2 Case 2: $\varsigma < \tau$

In this case, we note that the integrals in (64) and (65) are nonpositive, and so it follows that we have the upper bound

$$\rho(\varsigma, \tau) \leq \rho(\varsigma, \varsigma) \wedge \rho(\tau, \tau).\tag{71}$$

To derive a lower bound, note that

$$\mathbb{P}[\aleph_1(\nu) = \aleph_2(\varsigma)] \leq \rho(\nu, \varsigma) \leq \rho(\varsigma, \varsigma),\tag{72}$$

and so applying (72) to (64), we arrive at

$$\rho(\varsigma, \tau) \geq [1 + \varsigma - \tau] \rho(\varsigma, \varsigma).\tag{73}$$

Using a similar argument applied to (65), it can be shown that

$$\rho(\varsigma, \tau) \geq [1 + \varsigma - \tau] \rho(\tau, \tau).\tag{74}$$

Hence, by combining (73) and (74), we arrive at

$$\rho(\varsigma, \tau) \geq (1 + \varsigma - \tau) (\rho(\varsigma, \varsigma) \vee \rho(\tau, \tau)).\tag{75}$$

Thus, by combining (71) and (75), we arrive at the two sided bound

$$(1 + \varsigma - \tau) (\rho(\varsigma, \varsigma) \vee \rho(\tau, \tau)) \leq \rho(\varsigma, \tau) \leq (\rho(\varsigma, \varsigma) \wedge \rho(\tau, \tau))\tag{76}$$

Note that the lower bound (75) may also experience the same problems as the upper bound (69), resulting in a trivial lower bound in (76).

### 4.2.3 Some Comments on the Representation in Equation (64)

Consider again the representation of (64), which we write in the form

$$\rho(\varsigma, \tau) = \rho(\varsigma, \varsigma) + \int_{\tau}^{\varsigma} \mathbb{P}[\aleph_1(\nu) - \aleph_2(\varsigma) = 0] d\nu, \quad (77)$$

where  $\aleph_1(\cdot)$  and  $\aleph_2(\cdot)$  are independent Poisson random variables. Define the random variable

$$Z(\nu, \varsigma) = \aleph_1(\nu) - \aleph_2(\varsigma). \quad (78)$$

The difference of independent Poisson random variables has been studied extensively in the literature [Irwin 1937, Karlis and Ntzoufras 2003 and Skellam 1946], and is known as a Skellam distribution. This distribution has found application in the analysis of sports data [see Karlis and Ntzoufras 2003 and references contained therein]. Thus, the probability in the integral component of (77) is the probability that a Skellam distribution takes the value zero. Hence it may be possible to find some good approximations, and bounds, for  $\mathbb{P}[Z(\nu, \varsigma) = 0]$  in the literature. A preliminary review has found Gaussian approximations for this, but they were considered not to be useful for the applications considered in this report. Part of the problem is that the random variable  $Z(\nu, \varsigma)$  in (78) has mean  $\nu - \varsigma$  and variance  $\nu + \varsigma$ , so that a Gaussian approximation applied to  $\mathbb{P}[-1 < Z(\nu, \varsigma) < 1]$  will result in intractable integrals in (77). What is needed is an easily integrated approximation.

In further research in this area, the author plans to examine the literature more extensively, for work on the Skellam distribution, in order to improve the bounds of Subsection 4.2.

In the next Section we perform some numerical comparisons to investigate whether the new bounds (70) and (76) provide any improvement on the bounds in Subsection 4.1.

## 4.3 Numerical Comparison of Bounds

We are now in a position to compare the bounds of Subsection 4.2 to those in Subsection 4.1. Extensive numerical experiments showed that the new bounds of Subsection 4.2 can provide some improvements on the corresponding bounds in Subsection 4.1, but are not globally better. We consider a selection of cases to show the strengths and weaknesses of these new bounds.

Each figure includes a comparison of the bounds to an almost exact result. The latter has been obtained via recursive adaptive Simpson quadrature, with a tolerance of  $10^{-6}$ .

Figures 1 and 2 are plots of upper bounds for the case where  $\varsigma > \tau$ . In this case, we compare the upper bound (54), due to [Corazza and Ferrari 2002] to the new upper bound in (70). Note that the vertical scale is in natural logarithms. Each Figure contains two subplots. The first subplot shows the bounds as a function of  $\tau$ , for a fixed  $\varsigma$ , while the second subplot shows it as a function of  $\varsigma$ , for a fixed  $\tau$ . The legend used in the Figure refers to the bound of [Corazza and Ferrari 2002] as ‘C-F’, while ‘Wei’ refers to the bound in (70). The probability estimated from adaptive Simpson quadrature is referred to as

‘Exact’. As can be observed, the new upper bound provides no improvement on that due to [Corazza and Ferrari 2002]. Other numerical experiments provided no improvement on this result. This is most likely due to the fact that more work needs to be done to obtain a satisfactory estimate of the integral component of (77).

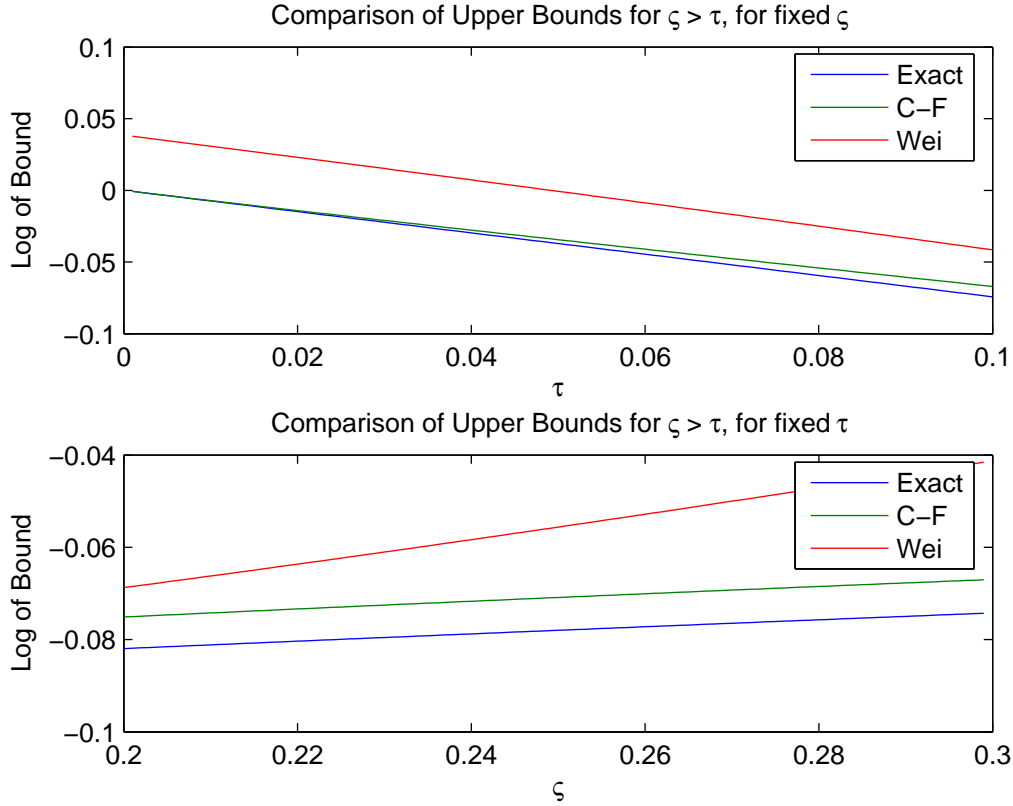


Figure 1: Comparison of upper bounds for the case where  $\zeta > \tau$ , showing the bounds (54) and (70) as functions of  $\tau$  and  $\zeta$ . ‘C-F’ refers to the upper bound due to Corazza and Ferrari, namely (54), while ‘Wei’ is the upper bound in (70). The vertical scale is in logarithms.

We now consider upper bounds for the case where  $\zeta < \tau$ . In this situation, we have four upper bounds to compare to the new upper bound in (76). Specifically, we are interested in how the upper bounds in (60)-(63) compare to (76). Two sets of  $(\zeta, \tau)$  parameters are considered. Figures 3 and 4 are for one case, while Figures 5 and 6 are a different case. Bounds in the Figures are abbreviated as ‘C-F’, for that due to [Corazza and Ferrari 2002], ‘S-A’, for [Simon and Alouini 2000], ‘Sim’ for [Simon 1998], ‘Chi’ for [Chiani 1999] and ‘Wei’ for that in (76). As before, the vertical scale is in logarithms. The first subplot of Figure 3 shows that the new bound (76) performs better than all others, with the exception of the upper bound (62), due to [Chiani 1999]. The second subplot shows a case where the new bound has superior performance. Figure 4 is the same as Figure 3, except the two worst performing bounds have been removed.

Figures 5 and 6 show the upper bounds (60)-(63) compared to (76), in a slightly different



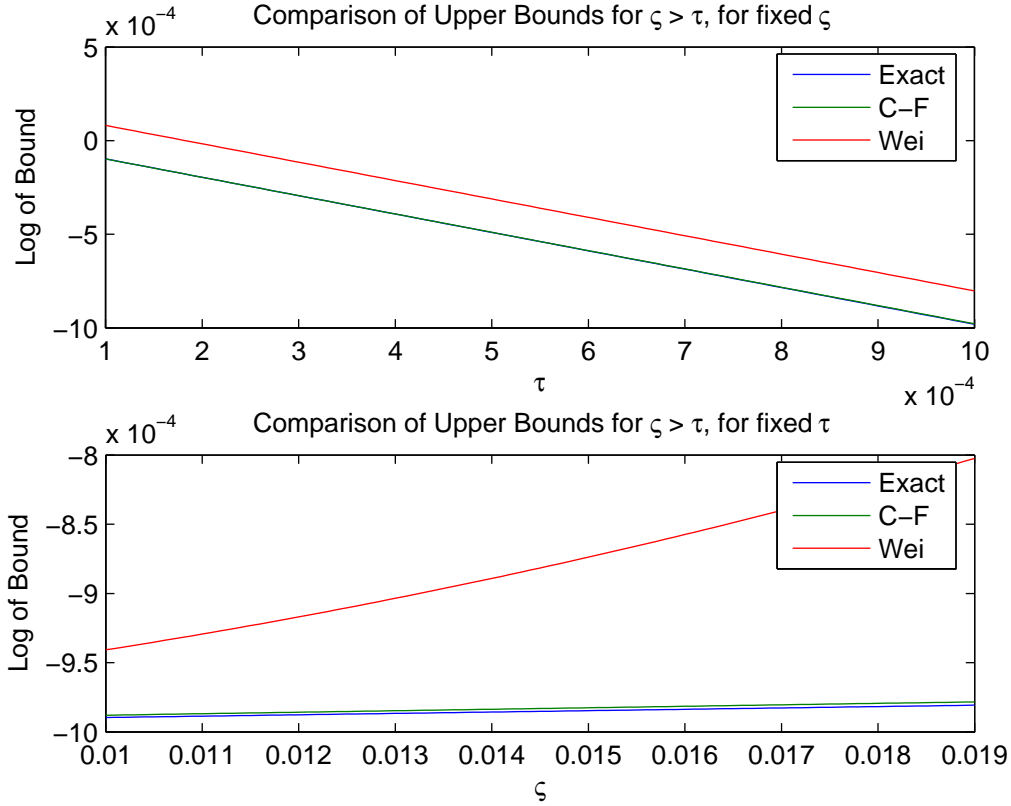


Figure 2: Similar plot to that of Figure 1, except on a different range of  $\zeta$  and  $\tau$  values.

scenario. In this example, the upper bound due to [Chiani 1999] has the best performance, while the upper bound in (76) tends to be very inaccurate. Figure 6 is the same as Figure 5, with the removal of the two worst performing bounds.

We now investigate lower bounds, beginning with the case where  $\zeta > \tau$ . Figures 7 and 8 are for one particular selection of  $(\zeta, \tau)$  parameters. The lower bounds investigated are those in (56)-(59), and the new lower bound in (70). The bounds are described in the legend using the same abbreviations as before. Although difficult to see, the exact values lie almost on the horizontal axis. Figures 7 and 8 shows that the new bound (70) performs well, and is only outperformed by the lower bound (56) due to [Corazza and Ferrari 2002].

The final scenario we consider is lower bounds in the case where  $\zeta < \tau$ . Figure 9 shows a plot of the lower bound components of (60)-(63), as well as the new lower bound in (76). The same naming conventions are employed in the legend, and the vertical scale is also in logarithms. The first subplot of Figure 9 is an example where the new lower bound in (76) has superior performance. The second subplot in Figure 9 shows the same effect, except the lower bound in (62) is slightly better for small values of  $\tau$ . Figure 10 is the same as Figure 9, except the two best performing bounds are compared.

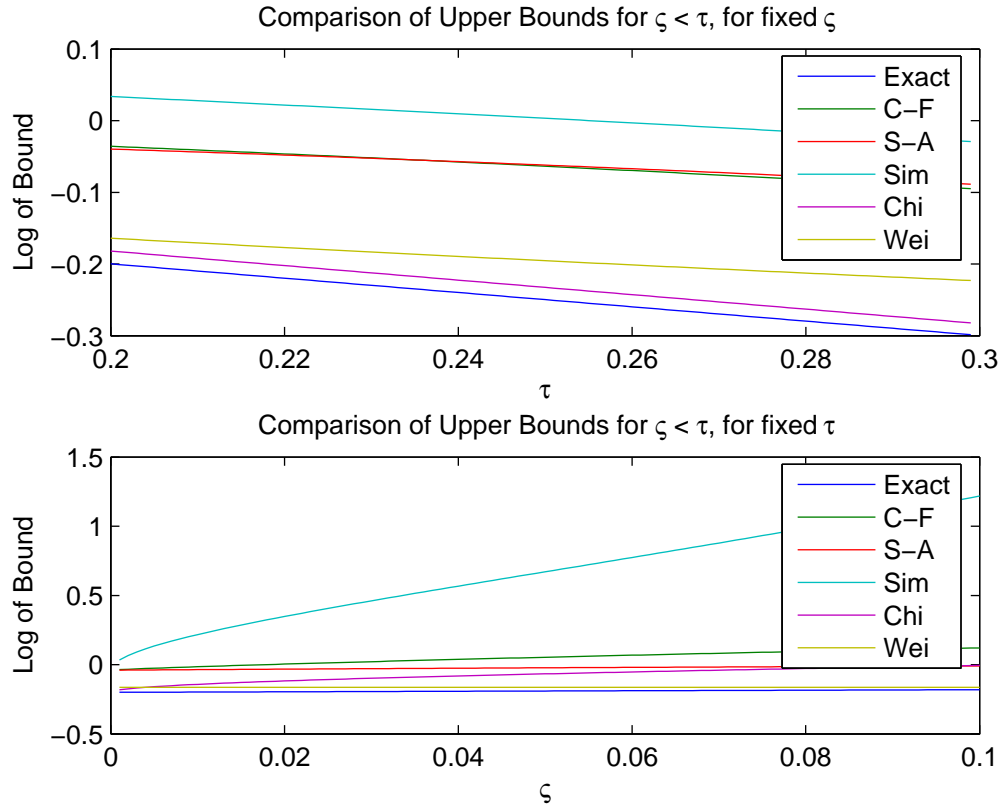


Figure 3: A comparison of upper bounds in the case where  $\varsigma < \tau$ . The new upper bound in (76), labelled 'Wei', is outperformed by the bound (62) due to Chiani, labelled as 'Chi'.

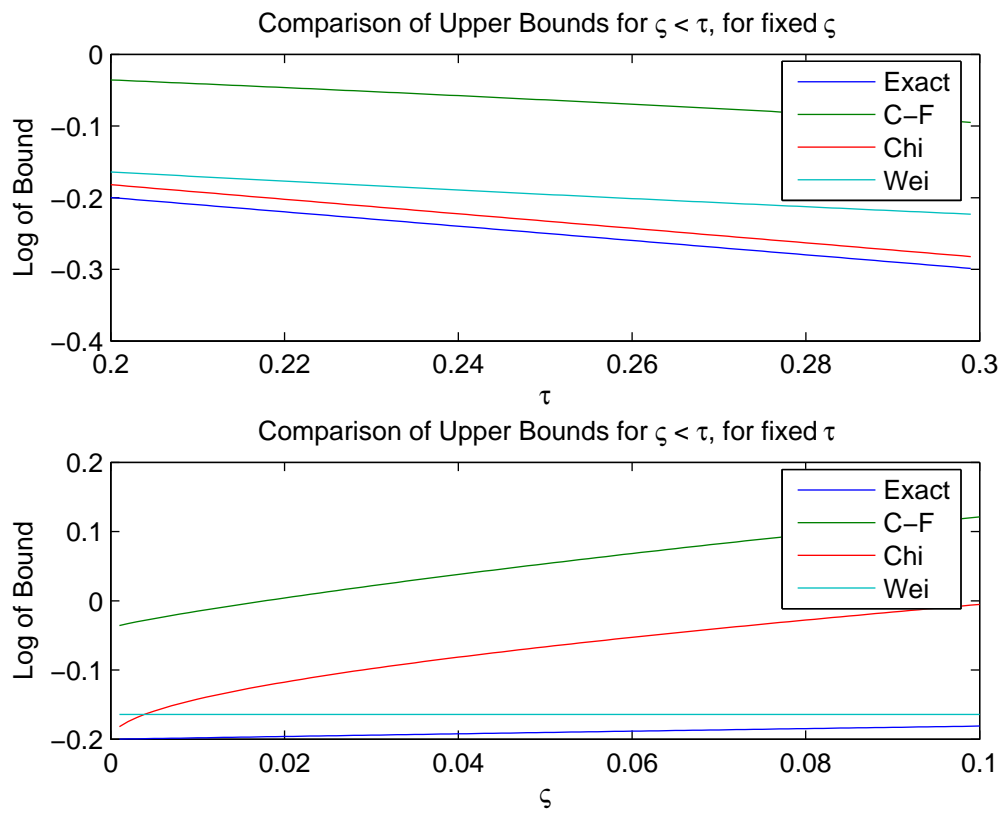


Figure 4: The same plot as for Figure 3, with the removal of the two worst performing bounds.

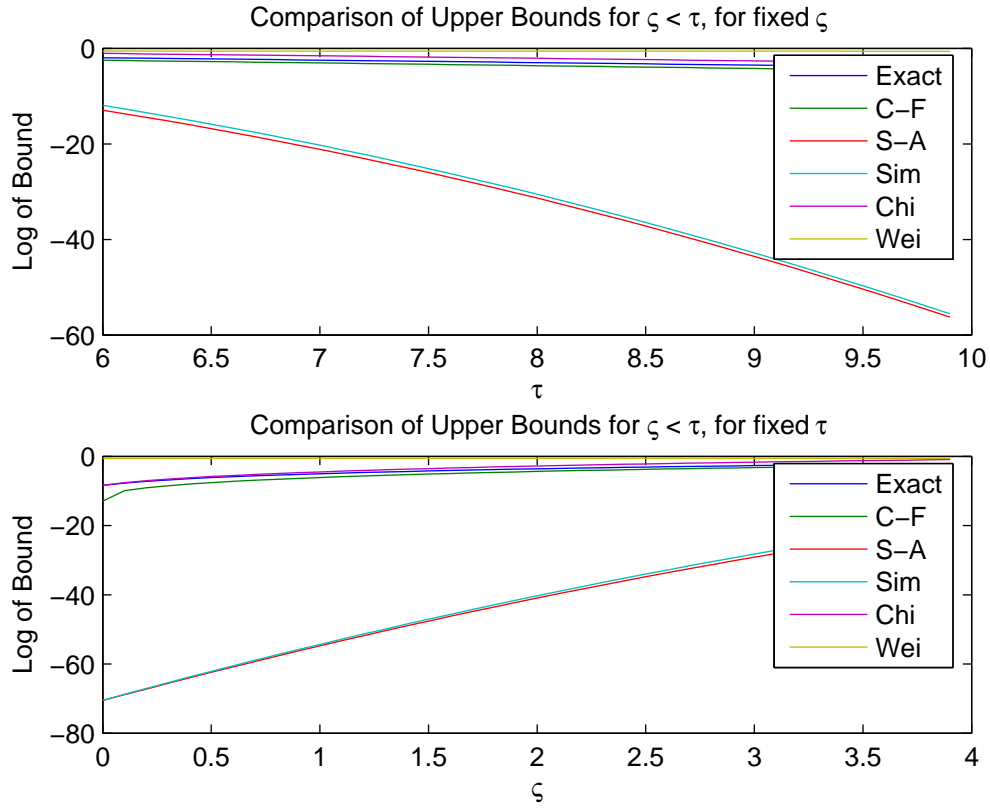


Figure 5: Comparison of upper bounds in the case where  $\zeta < \tau$ , but over a different range of  $\zeta$  and  $\tau$  parameters. In both subplots, it is clear that the bounds 'Sim', due to Simon (63) and 'S-A', due to Simon and Alouini (61) are unsatisfactory. Figure 6 shows the same plot with these two upper bounds removed.

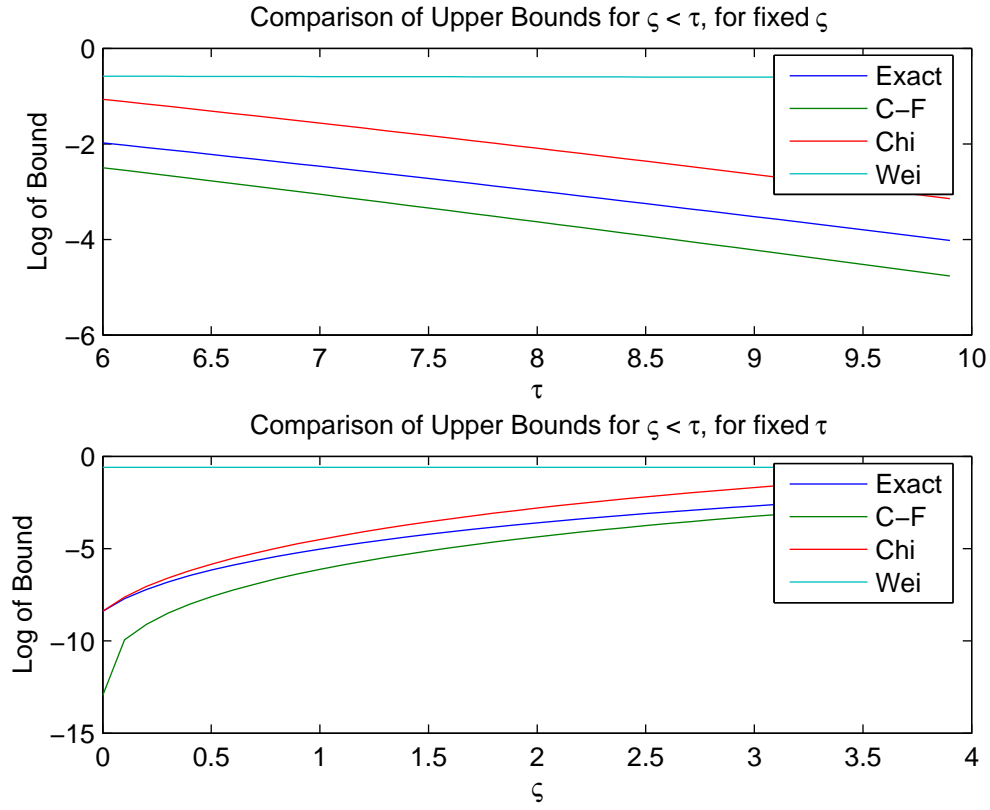


Figure 6: This is the same as Figure 5, except the two worst performing bounds have been removed. It is now clear that the bound labelled 'C-F', due to Corazza and Ferrari (60) is also unsatisfactory. The bound due to Chiani (62) performs the best, while the new upper bound in (76), labelled as 'Wei', is the second best.

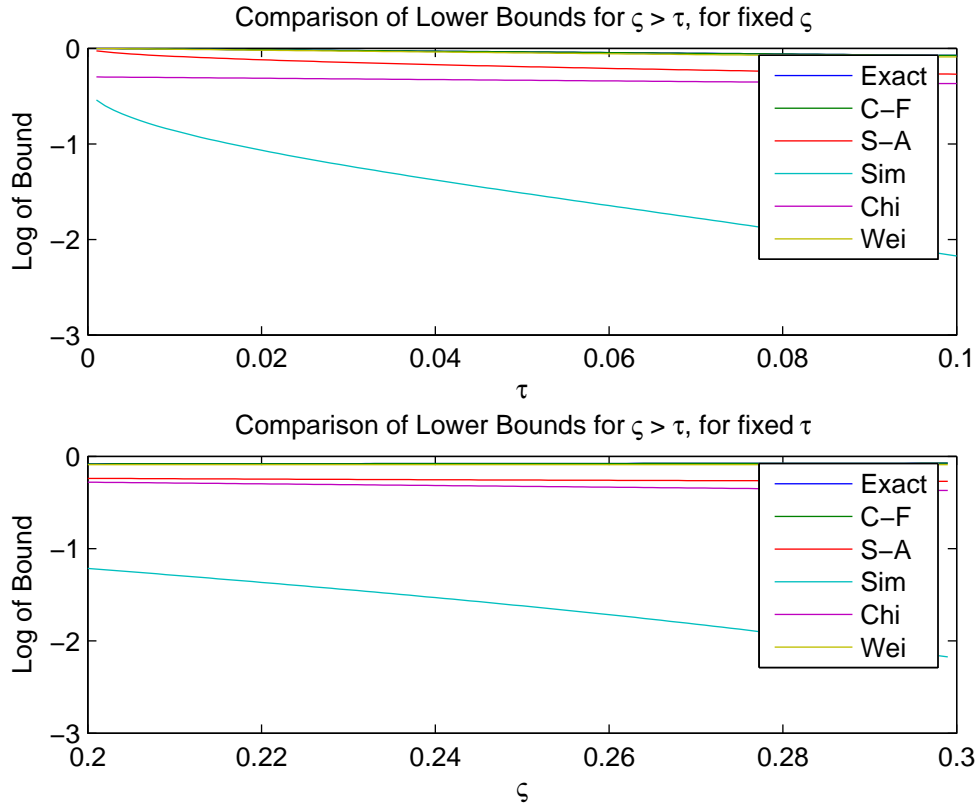


Figure 7: Plots of lower bounds in the case where  $\zeta > \tau$ . The same labelling convention has been employed as in previous figures. The exact value, labelled as 'Exact', is almost on the horizontal axis in each subplot. It is clear that the new bound (70) is performing very well. See Figure 8 for clarification of this.

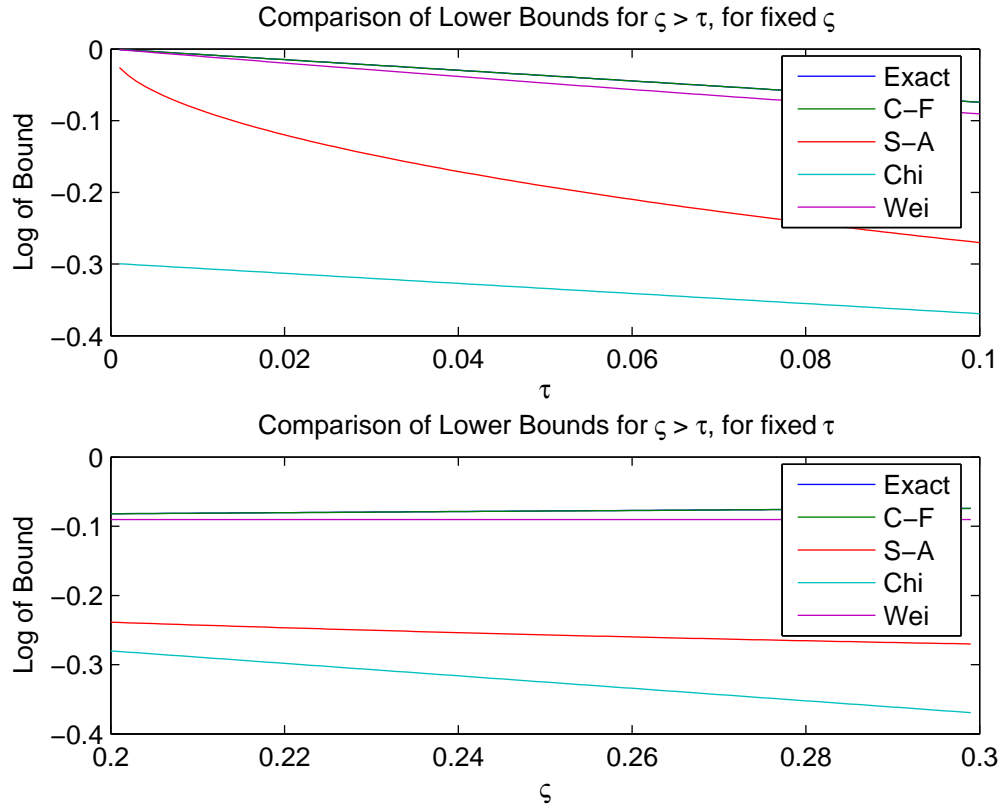


Figure 8: The same plot as for Figure 7, except with the removal of the bound (59) due to Simon. As for Figure 7, the exact value is almost on the horizontal axis. The best bound is that due to Corazza and Ferrari (56), while the new lower bound in (70), namely ‘Wei’, is second best.

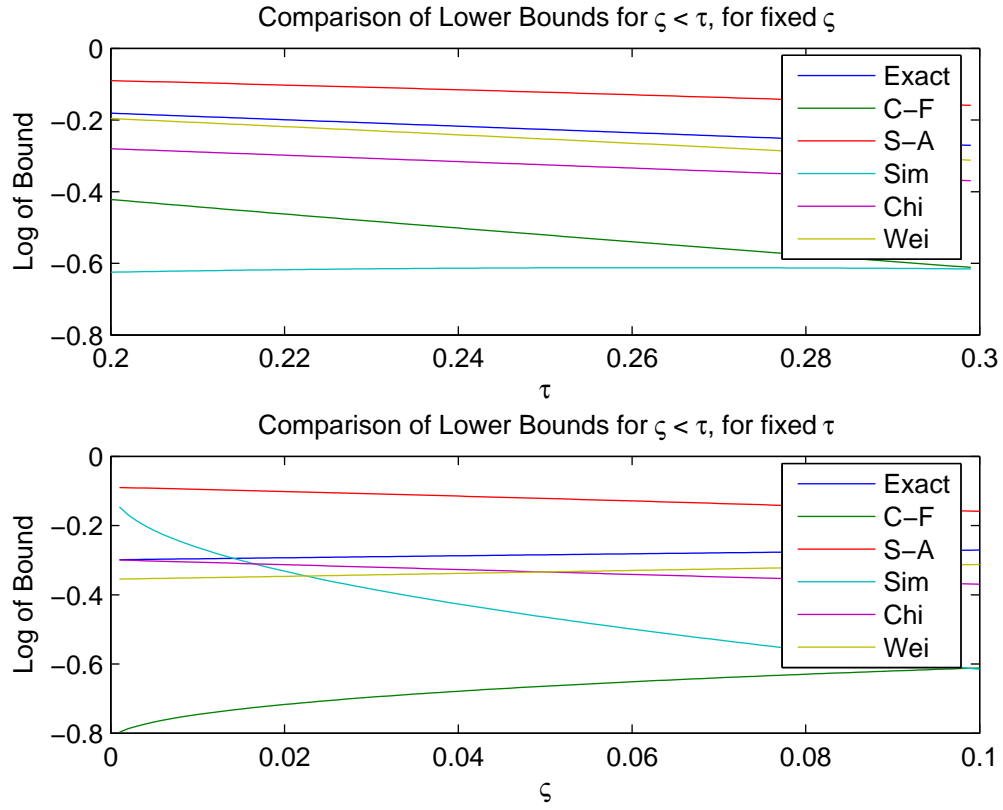


Figure 9: Lower bounds for the case where  $\varsigma < \tau$ . This is an example where the new lower bound in (76) has very good performance. The only bound that is comparable to the new lower bound is that due to Chiani (62). The latter is better in the second subplot when  $\varsigma$  is very small.



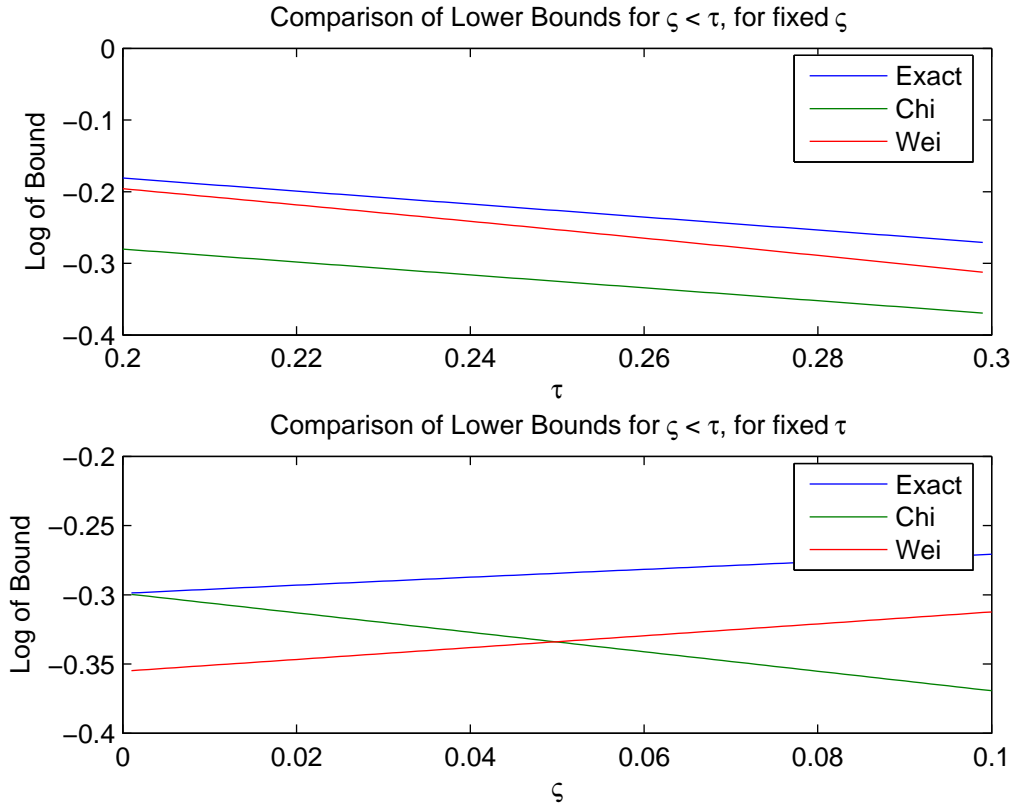


Figure 10: The same plot as in Figure 9, with the inclusion of the Chiani lower bound in (62), and the new lower bound in (76), and the exclusion of the other lower bounds. The first subplot shows the improvement more clearly, while the second subplot shows there is a point where the better bound switches from the Chiani one to the new lower bound.

## 5 Conclusions and Future Directions

This report introduced a number of new results. A Poisson association with the Marcum Q-Function, and associated detection probabilities, was derived, extending the case considered in [Weinberg and Kyprianou 2005]. Stein's method was used to produce a new expression for the distributional differences of a pair of Poisson random variables. This permitted the construction of two Volterra integral equations, whose solution was the detection probability under consideration. These new representations were used to construct lower and upper bounds on the single pulse probability of detection. Numerical comparisons to other bounds in the literature showed there are cases where these new bounds provided improvements.

It is believed that better global lower and upper bounds could be achieved through further analysis of these Volterra integral equations. Specifically, there may be properties known about such equations that will lead to new ways of estimating the Marcum Q-Function, and associated detection probabilities. As pointed out in the report, it may be possible to estimate the detection probability (15) using a Picard iteration scheme applied to (43). Also, using estimates of the Skellam distribution's zero probability, applied to (77), may also produce improved bounds. This will be the focus of subsequent research in this area.

## 6 Acknowledgements

Thanks are due to the Task Sponsor, Commander Surveillance & Response Group, RAAF Williamtown for supporting this research. The author would like to thank Dr Tom Winchester for reviewing the report, and drawing the author's attention to the connection between the Marcum Q-Function and detection probabilities. Also, Peter Lawrence of WSD, vetted the report, and I thank him for a fast and useful review. Finally, thanks are also due to Dr Aihua Xia, of the University of Melbourne, for useful discussions on the comparison of Poisson distributions.

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## Appendix A: Proof of Equation (53)

Suppose  $X$  and  $Y$  are independent and identically distributed Poisson random variables with mean  $\lambda$ . Then, with reference to the expansion (18),

$$\mathbb{P}(X = Y) = \sum_{k=0}^{\infty} \left( \frac{e^{-\lambda} \lambda^k}{k!} \right)^2 = e^{-2\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(k!)^2} = e^{-2\lambda} I_0(2\lambda). \quad (\text{A.1})$$

Since  $X$  and  $Y$  are statistically identical, it follows that

$$\mathbb{P}(X < Y) \equiv \mathbb{P}(X > Y). \quad (\text{A.2})$$

Thus, (A.2) implies that

$$\mathbb{P}(X \neq Y) = 2\mathbb{P}(X < Y). \quad (\text{A.3})$$

Hence, with an application of (A.1) and (A.3), it follows that

$$\begin{aligned} \mathbb{P}(X \leq Y) &= \mathbb{P}(X = Y) + \mathbb{P}(X < Y) \\ &= e^{-2\lambda} I_0(2\lambda) + \frac{1}{2} [1 - e^{-2\lambda} I_0(2\lambda)] \\ &= \frac{1}{2} [1 + e^{-2\lambda} I_0(2\lambda)]. \end{aligned} \quad (\text{A.4})$$

The desired result now follows by an application of (A.4) to (23) in the case where  $n = 1$ .  
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4. AUTHOR Graham V. Weinberg			5. CORPORATE AUTHOR Defence Science and Technology Organisation PO Box 1500 Edinburgh, South Australia, Australia 5111		
6a. DSTO NUMBER DSTO-RR-0304		6b. AR NUMBER AR-013-559		6c. TYPE OF REPORT Research Report	
7. DOCUMENT DATE December, 2005					
8. FILE NUMBER 2005/1063348/1	9. TASK NUMBER AIR 04/206	10. SPONSOR CDR SRG	11. No OF PAGES 38	12. No OF REFS 36	
13. URL OF ELECTRONIC VERSION <a href="http://www.dsto.defence.gov.au/corporate/reports/DSTO-RR-0304.pdf">http://www.dsto.defence.gov.au/corporate/reports/DSTO-RR-0304.pdf</a>			14. RELEASE AUTHORITY Chief, Electronic Warfare and Radar Division		
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